Godement-Jacquet zeta integrals on GL(2, 𝔽)

Taku ISHII*

ABSTRACT: We evaluate the complex archimedean part of Godement-Jacquet zeta integrals for GL(2). We explicitly give test vectors for the zeta integrals, that is we show the coincidence of the zeta integrals and the Langlands L-factors by direct computation of zeta integrals.

Keywords: automorphic forms, standard L-functions, zeta integrals, principal series representations

Introduction

The book of Godement and Jacquet [1] is one of a milestone in the theory of automorphic L-functions. The standard L-functions on the general linear group GL(n) are defined, and some fundamental properties of these L-functions are established via integral representations. Because of their importance, the standard L-functions on GL(n) are also called the principal L-functions ([3]). Since grüssencharacter can be regarded as an automorphic representation of GL(1), the work of Godement and Jacquet is as a generalization of Iwasawa-Tate theory.

In this paper we explicitly compute complex local zeta integral of Godement and Jacquet as in Tate's thesis. More precisely, we give an example of “test vector” for the zeta integrals in case of the principal series representation of GL(2, 𝔽).

1. Review of Godement-Jacquet local theory

We review the local (archimedean) theory of Godement-Jacquet [1]. Let 𝐹 be an archimedean local field. Set 𝐴 = 𝑀(𝑛, 𝐹) and 𝐺 = 𝐴× = GL(𝑛, 𝐹). Fix a maximal compact subgroup 𝐾 of 𝐺 by 𝐾 = 𝐟(𝑛) if 𝐹 = ℜ and 𝐾 = 𝑈(𝑛) if 𝐹 = ℂ. We denote by || the modulus of 𝐹 defined by ||| 𝑥| = | 𝑥 | and | 𝑥 | = 𝑥 2 where | | is the ordinary absolute value.

Let (π, 𝑉) be an irreducible admissible representation of 𝐺, and (π′, 𝑉′) be its contragredient representation. We denote by ⟨ , ⟩ the canonical pairing on 𝑉 × 𝑉′. For 𝑣 ∈ 𝑉 and 𝑣′ ∈ 𝑉′, we define the function 𝛽𝑣,𝑣′ on 𝐺 by 𝛽𝑣,𝑣′(𝑔) = ⟨ 𝜋(𝑔)𝑣, 𝑣′ ⟩(𝑔 ∈ 𝐺). We call 𝐶(𝜋) = ℂ−spa{𝛽𝑣,𝑣′ | 𝑣 ∈ 𝑉, 𝑣′ ∈ 𝑉′} the space of matrix coefficient of 𝜋. For 𝛽 ∈ 𝐶(𝜋), set 𝛽′(𝑔) = 𝛽(𝑔−1). Then we have 𝛽′ ∈ 𝐶(𝜋).

We denote by 𝐿(𝐴) be the space of Schwartz-Bruch function on 𝐴. Let 𝐿0(𝐴) be the (dense)

subspace of 𝐿(𝐴) consisting of the functions of the form

\[
\left\{
\begin{array}{ll}
P(𝑥_{ij}) \exp(-π \text{tr}(𝑡𝑥)) & \text{if } 𝐹 = ℜ; \\
P(𝑥_{ij}) \exp(-2π \text{tr}(𝑡𝑥)) & \text{if } 𝐹 = ℂ,
\end{array}
\right.
\]

for 𝑥 = (𝑥_{ij}) ∈ 𝑀(𝑛, 𝐹), where 𝑃 is a polynomial in 𝑥_{ij}. Let 𝜉 be the nontrivial additive character of 𝐹 defined by

\[
\left\{
\begin{array}{ll}
𝜉(𝑥) = \exp(2π𝑖𝑥) & \text{if } 𝐹 = ℜ \text{ and } 𝑥 ∈ ℜ; \\
𝜉(𝑥) = \exp(2π𝑖(𝑥 + 𝑥)) & \text{if } 𝐹 = ℂ \text{ and } 𝑥 ∈ ℂ.
\end{array}
\right.
\]

For 𝜙 ∈ 𝐿0(𝐴), we define the Fourier transform 𝜙̂ of 𝜙 with respect to 𝜉 by

\[
\hat{𝜙}(𝑥) = \int_{𝐴} 𝜙(𝑦)𝜉(\text{tr}(𝑥𝑦))d_𝜙 𝑦,
\]

where 𝑑_𝜙𝑦 is the self-dual Haar measure on 𝐴. When 𝜙 ∈ 𝐿0(𝐴), then we have 𝜙̂ ∈ 𝐿0(𝐴).

For 𝑠 ∈ ℂ, 𝛽 ∈ 𝐶(𝜋) and 𝜙̂ ∈ 𝐿0(𝐴), the archimedean Godement-Jacquet zeta integral is defined by

\[
\zeta(s, 𝜙, 𝛽) = \int_{𝐴} 𝜙̂(𝑔)𝜉(\beta(𝑔))\, dg.
\]

Here is a main archimedean local result of [1] and [3].

Proposition 1. ([1], [3]) Let 𝜋 be an irreducible admissible representation. The archimedean zeta integral 𝑍(𝑠, 𝜙, 𝛽) (𝜙 ∈ 𝐿0(𝐴), 𝛽 ∈ 𝐶(𝜋)) defined above has the following properties.

1. There exists ℠0 ∈ 𝑅 such that 𝑍(𝑠, 𝜙, 𝛽) converges absolutely for Re(𝑠) > ℠0.
2. There exists an L-factor such that for each 𝜙 and 𝛽, the ratio 𝑍(𝑠, 𝜙, 𝛽)/𝐿(𝑠, 𝜋) is a polynomial in 𝑠. Moreover there exists a finite set (𝜙1, 𝛽1) ∈ 𝐿0(𝐴) × 𝐶(𝜋), such that 𝐿(𝑠, 𝜋) = \sum_{(𝜙, 𝛽) ∈ (𝜙1, 𝛽1)} 𝑍(𝑠, 𝜙, 𝛽).
3. There exists an ε-factor 𝜖(𝑠, 𝜋, 𝜉) ∈ ℂ× such that the local functional equation

\[
\frac{Z(1−𝑠, 𝜙̂, 𝛽′)}{L(1−𝑠, 𝜋)} = 𝜖(𝑠, 𝜋, 𝜉)\frac{Z(s, 𝜙, 𝛽)}{L(s, 𝜋)}
\]

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holds. (4) The L- and ε-factors defined above coincide with the L- and ε-factors determined by the local Langlands correspondence.

Let \( k \) be a number field, and \( \pi = \pi_v \), \( \pi_v \) be a cuspidal automorphic representation of \( \text{GL}(n, A_k) \). Here \( A_k \) is the adele of \( k \). Since \( \pi \) is generic, the local representation \( \pi_v \) is also generic. Then, for archimedean place \( v \), \( \pi_v \) is isomorphic to an irreducible generalized principal series representation of \( \text{GL}(n, \mathbb{R}) \) induced from the parabolic subgroup corresponding to the partition \( (2, \cdots, 2, 1, \cdots, 1) \) when \( v \cong \mathbb{R} \), and isomorphic to an irreducible principal series representation of \( \text{GL}(n, \mathbb{C}) \) when \( v \cong \mathbb{C} \).

Our aim of this paper is to give a pair \( (\Phi, \beta) \) explicitly such that \( Z(s, \Phi, \beta) = L(s, \pi) \) when \( n = 2 \). Since the case of \( F = \mathbb{R} \) is discussed in [2], we consider the case where \( \pi \) is the principal series representation of \( \text{GL}(2, \mathbb{C}) \).

2. Representation theory of \( \text{GL}(2, \mathbb{C}) \)

Hereafter let \( n = 2 \), \( F = \mathbb{C} \), and \( G = \text{GL}(2, \mathbb{C}) \). We first recall a representation theory of \( K = \text{U}(2) \), that is complex analytic finite dimensional representation of \( \text{GL}(2, \mathbb{C}) \). For \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 \) with \( \lambda_1 \geq \lambda_2 \) we denote by \( (\tau_\lambda, V_\lambda) \) the irreducible finite dimensional representation of \( \text{GL}(2, \mathbb{C}) \) with the highest weight \( \lambda \). The representation space \( V_\lambda \) consists of a homogeneous polynomial \( f(x_1, x_2) \in \mathbb{C}[x_1, x_2] \) with degree \( \lambda_1 - \lambda_2 \), and the action of \( \text{GL}(2, \mathbb{C}) \) is given by
\[
(\tau_\lambda(f))(x_1, x_2) = (\det g)^{\lambda_2} f((x_1, x_2)g), \quad g \in G.
\]

We take a standard basis
\[
\{v_\lambda^p : = x_1^p x_2^{\lambda_1 - \lambda_2 - p} | 0 \leq p \leq \lambda_1 - \lambda_2 \}.
\]
Then the inner product defined by
\[
\langle v_\lambda^p, v_\lambda^q \rangle = \delta_{p, q} (\lambda_1 - \lambda_2)^{-1}
\]
is \( K \)-invariant.

We review the principal series representations of \( G \). For an integer \( d \) and a complex number \( \nu \), define a character \( \chi_{\nu, d}(t) = |t|^{\nu d} \overline{t}(t/|t|)^d \) \((t \in \mathbb{C}^\times)\). For \( d = (d_1, d_2) \in \mathbb{Z}^2 \) and \( \nu = (\nu_1, \nu_2) \in \mathbb{C}^2 \), we define the induced representation \( (\pi_{\nu, d}, V_{\nu, d}) \) of \( G \) by
\[
V_{\nu, d} = \{ f : G \rightarrow \mathbb{C} \mid f((a_1, a_2)(g) \}
\]
\[
V_{\nu, d} = \chi_{\nu_1, d_1}(a_1) \chi_{\nu_2, d_2}(a_2) |a_1/ a_2|^{1/2} f(g), \forall g \in G, \}
\]
on which \( G \) acts by right translation. We call \( \pi_{\nu, d} \) the principal series representation of \( G \). Since \( \pi_{(\nu_1, \nu_2), (d_1, d_2)} \cong \pi_{(\nu_2, \nu_1), (d_2, d_1)} \), hereafter we assume that \( d_1 \geq d_2 \). Under this assumption we have
\[
\pi_{(\nu_1, \nu_2), (d_1, d_2)} |K = \bigoplus_{m \geq 0} \tau_{(d_1 + m, d_2 - m)}.
\]
We note that the contragredient representation of \( \pi_{\nu, d} \) is isomorphic to \( \pi_{-\nu, d^*} \) with \( \nu^* = (-\nu_2, -\nu_1) \) and \( d^* = (-d_2, -d_1) \).

According to the local Langlands correspondence the L-factor \( L(s, \pi_{\nu, d}) \) for the principal series is given by
\[
L(s, \pi_{\nu, d}) = \prod_{i=1, 2} \Gamma(\chi + \nu_i + |d_i|/2).
\]

3. Calculus of archimedean zeta integrals

Let us compute \( Z(s, \Phi, \beta) \). The matrix coefficient for \( \pi \) can be written by the integral
\[
\beta f_1, f_2 (g) = \int_K f_1(kg) f_2(k) dk, \quad g \in G,
\]
where \( f_1 \in \pi = \pi_{\nu, d} \) and \( f_2 \in \tilde{\pi} \). Here the Haar measure \( dk \) on \( K \) is normalized as \( \int_K dk = 1 \).

Using the Iwasawa decomposition \( G = NAK \) with
\[
N = \{ (\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) | x \in \mathbb{C} \},
\]
\[
A = \{ (\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}) | a_1, a_2 > 0 \},
\]
we normalize the Haar measure \( dg \) on \( G \) by
\[
dg = nd x \overline{a}_d dk = a_1^{-2} \cdot d_{\mathbb{C}^\times} 2a_1/ a_2 dk
\]
for \( g = (\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) k \). Here \( d_{\mathbb{C}^\times} = 2dx_1 dx_2 (x = x_1 + \sqrt{-1}x_2) \) with the ordinary Lebesgue measure \( dx_i \) on \( \mathbb{R} \).

Then we have
\[
Z(s, \Phi, \beta f_1, f_2)
\]
\[
= \int_G \Phi(g) \int_K f_1(kg) f_2(k) |\det g|^{s+1/2} dk dg
\]
\[
= \int_G \Phi(k^{-1} g) f_1(kg) f_2(k) |\det g|^{s+1/2} dk dg
\]
\[
= \int_{N \times K} f_1(nak') f_2(k) \Phi(k^{-1} nak')
\]
\[
\times |\det a|^{s+1/2} dk dk' d\overline{a}_d d\overline{a}_d a.
\]
If we denote by \( \Phi(g) = P_\Phi(g) \exp(-2\pi tr(\overline{g} g)) \)
\((g \in G)\) with polynomial \( P_\Phi \) on \( A \), then we have
\[
Z(s, \Phi, \beta f_1, f_2)
\]
\[
= \int_0^\infty \int_0^\infty I(a_1, a_2, x) \cdot a_1^{2(s+\nu_1)} a_2^{2(s+\nu_2)}
\]
\[
\times \exp(-2\pi (a_1^2 + a_2^2 + x\overline{x})) \overline{a}_d 2 \overline{a}_d d_{\mathbb{C}^\times},
\]
where
\[
I(a_1, a_2, x)
\]
\[
= \int_K f_1(k') f_2(k) P_\Phi(k^{-1} \begin{pmatrix} a_1 & x \\ 0 & a_2 \end{pmatrix}) dk dk'.
\]
Therefore if we can choose \((f_1, f_2, \Phi)\) such that
\[
I(a_1, a_2, x) = a_1^{|d_1|} a_2^{d_2},
\]
the formulas
\[
\begin{align*}
\int_{C} \exp(-2\pi x\bar{x}) \, dC \, x &= 1, \\
\int_{0}^{\infty} \exp(-2\pi a^2) \, a^{2^2/a} \, d\nu(a) &= \Gamma_C(s)
\end{align*}
\]
implies that
\[
Z(s, \Phi, \beta_{f_1, f_2}) = \prod_{i=1,2} \Gamma_C(s + \nu_i + \frac{|d_i|}{2}).
\]

Here is our main result.

**Theorem 2.** When \(\pi\) is isomorphic to the irreducible principal series representation \(\pi_{\nu, d}\) with \(d_1 \geq d_2\), we take \(f_1 \in \pi, f_2 \in \pi \cong \pi_{\nu, d}\) and \(\Phi \in S(A)\) as follows. For arbitrary \(l\) and \(m\) with \(0 \leq l, m \leq d_1 - d_2\), set
\[
\begin{align*}
f_1(k) &= (d_1 - d_2 + 1) \\
&\times \langle (\tau_\delta(k)v^{d_1}, v_{d_1 - d_2}) \rangle,
\end{align*}
\[
\begin{align*}
f_2(k) &= (-1)^m (d_1 - d_2 + 1) \\
&\times \langle (\tau_{d_2}(k)v^{d_2}, v_{d_2 - d_1}) \rangle,
\end{align*}
\]
for \(k \in K\). Choose \(\delta = (\delta_1, \delta_2)\), \(\delta' = (\delta'_1, \delta'_2)\) as follows.

<table>
<thead>
<tr>
<th>(d_1 \geq d_2 \geq 0)</th>
<th>(d_1 \geq 0 \geq d_2)</th>
<th>(0 \geq d_1 \geq d_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\delta)</td>
<td>(d_1, d_2)</td>
<td>((d_1, 0))</td>
</tr>
<tr>
<td>(\delta')</td>
<td>((d_2, 0))</td>
<td>((-d_2, -d_1))</td>
</tr>
</tbody>
</table>

We define \(\Phi(g) = P_\Phi(g) \exp(-2\pi \text{tr}(g'g^{-1}))\) by
\[
P_\Phi(g) = \sum_{\nu, p, q, d, q'} \langle \tau_\delta(g)v_{d_1}^{\nu}, v_1 \rangle \langle \tau_{d_2}(g)v_{d_2}^{\nu}, v_2 \rangle \\
\times \langle (v_1^{\nu}, q_1^{\nu}, v_1^{\nu}, q_1^{\nu}) \rangle \langle (v_2^{\nu}, q_2^{\nu}, v_2^{\nu}, q_2^{\nu}) \rangle^{-1},
\]
where \(\bar{g} = (\det(g))^{-1}g^{-1}\) and \(p, q, p', q'\) run through such that \(p + p' = l, q + q' = m, 0 \leq p, q \leq \delta_1 - \delta_2\), and \(0 \leq p', q' \leq \delta'_1 - \delta'_2\). Then we have
\[
Z(s, \Phi, \beta_{f_1, f_2}) = L(s, \pi).
\]

**Remark 1.** When \(d_1 \geq d_2 \geq 0\), we have
\[
P_\Phi(g) = \langle \tau_\delta(g)v_{d_1}^{\nu}, v_{d_1} \rangle \langle \tau_{d_2}(g)v_{d_2}^{\nu}, v_{d_2} \rangle^{-1}.
\]

**Proof.** Let us compute the integral \(I(a_1, a_2, x)\). Set \(a = (a_0, a_1, a_2)\), \(\alpha = (\alpha_0, \alpha_1, \alpha_2)\) and use the notation
\[
\eta_{d_1, d_2} = (\delta_1, \delta_2), \quad \nu(d_1, d_2) = (\delta_1, \delta_2).
\]

We have
\[
P_\Phi(k^{-1}b^k) = \sum_{\nu, \eta, p, q} \eta_{p, q}^{d_1, d_2} \langle \tau_\delta(k^{-1}b^k)v_1^{\nu}, v_1 \rangle \langle \tau_{d_2}(k^{-1}b^k)v_2^{\nu}, v_2 \rangle \\
\times \langle \eta_{p, q}^{d_1, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle^{-1},
\]

using
\[
\tau_\delta(g)v_{d_1}^{\nu} = \sum_{0 \leq \rho \leq \delta_1 - \delta_2} \langle \tau_\delta(g)v_1^{\nu}, v_1^{\nu} \rangle \langle \tau_{d_2}(g)v_{d_2}^{\nu}, v_2^{\nu} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle^{-1},
\]
we get
\[
P_\Phi(k^{-1}b^k) = \langle \tau_\delta(k)^{d_1, d_2} \rangle \langle \tau_{d_2}(k)^{d_1, d_2} \rangle \\
\times \sum_{\nu, \eta, p, q} \langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \\
\times \langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle^{-1},
\]
\[
\times \langle \eta_{p, q}^{d_1, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle^{-1},
\]
where \(r, r', s, s'\) run through such that \(0 \leq r, s \leq \delta_1 - \delta_2, 0 \leq r', s' \leq \delta'_1 - \delta'_2\). Now let us consider the sum over \(p, p', q, q'\).

**Lemma 3.** For \(k \in K\), we have
\[
\sum_{\nu, \eta, p, q} \langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \\
\times \langle \eta_{p, q}^{d_1, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle^{-1},
\]
the left hand side of (2) becomes
\[
\langle \tau_{\delta_1}(k)^{d_1, d_2} \rangle \langle \tau_{\delta_2}(k)^{d_1, d_2} \rangle \\
\times \sum_{\nu, \eta, p, q} \langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \\
\times \langle \eta_{p, q}^{d_1, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle^{-1},
\]
\[
\times \langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle^{-1}.\]

The last term \(\sum_{\nu, \eta, p, q, \cdots} \cdots\) can be written as
\[
\frac{1}{i!(r-i)!}\langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \\
\times \frac{1}{i!(r'-i')!}\langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle^{-1},
\]

\[
\times \frac{1}{i!(r-i)!}\langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle - \frac{1}{i!(r-i)!}\langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \\
\times \frac{1}{i!(r'-i')!}\langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle^{-1}.
\]

Thanks to the identity
\[
\sum_{p} \frac{1}{p!(A-p)!}(B-p)! \langle C+p \rangle!
\]
the left hand side of (2) equals to
\[
\langle \tau_{\delta_1}(k)^{d_1, d_2} \rangle \langle \tau_{\delta_2}(k)^{d_1, d_2} \rangle^{-1} \\
\times \sum_{\nu, \eta, p, q} \langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \\
\times \frac{1}{i!(r-i)!}\langle \eta_{\delta_1}^{d_1, d_2} \rangle \langle \eta_{\delta_2}^{d_2, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle \langle \eta_{p, q}^{d_1, d_2} \rangle^{-1}.
\begin{align*}
&\times d_{\delta_1+\delta'_1+\delta'_2+\delta'_3+\delta'_4-l-(r+r')}+(i+i') \\
&\times i\Gamma^2(r-i)(r'-i')!(l-i'+i'!)
\times (\delta_1-\delta_2+\delta'_1+\delta'_2-l-r-r'+i+i')!
= (\det k)^{\delta_2+\delta'_2}r^l r' \delta_1-\delta_2+\delta'_1-\delta'_2-l-r-r'+i+i')!
\times \sum_j a_j^b r^{r'-j-l-j} d_{\delta_1-\delta_2+\delta'_1-\delta'_2-l-r-r'+j}
\times (l-j)(\delta_1-\delta_2+\delta'_1-\delta'_2-l-r-r'+j))!
\times \sum_i l_{i}(j-i)!r(r-i)(r'-j+1)!,
\end{align*}

By using (3) again, the above becomes
\begin{align*}
&\times a^b r^{r'-j-l-j} \delta_1-\delta_2+\delta'_1-\delta'_2-l-r-r'+j)+j
= c_{\delta_2+\delta'_2}^{\delta_1+\delta'_1}(\tau_\delta(k) v_j^{\delta_1+\delta'_1}, v_j^{\delta_2+\delta'_2}).
\end{align*}

In view of the relation
\begin{align*}
\langle \tau_\delta(k) v_j^{\delta_1}, v_j^{\delta_2} \rangle = \langle \tau_\delta(k) v_j^{\delta_1}, v_j^{\delta_2} \rangle,
\end{align*}
we arrive at the identity (2). \hfill \Box

Now we return to the evaluation of $I(a_1, a_2, x)$. Lemma 3 leads us that
\begin{align*}
P_\Phi(k^{-1} b k')
= c_{\delta_2+\delta'_2}^{\delta_1+\delta'_1} \sum_{r,r',s'} c_{\delta_2+\delta'_2}^{\delta_1+\delta'_1} c_{\delta_2+\delta'_2}^{\delta_1+\delta'_1} c_{\delta_2+\delta'_2}^{\delta_1+\delta'_1}
\times (\det k)^{\delta_1+\delta'_1} (\tau_\delta(k) v_j^{\delta_1}, v_j^{\delta_2+\delta'_2+\delta'_3}) \times (\det k')^{\delta_2+\delta'_2} (\tau_\delta(k) v_j^{\delta_1}, v_j^{\delta_2+\delta'_2+\delta'_3}) \times (\tau_\delta(b) v_j^{\delta_1}, v_j^{\delta_2}) \times (\tau_\delta(b) v_j^{\delta_1}, v_j^{\delta_2}) \times (\tau_\delta(b) v_j^{\delta_1}, v_j^{\delta_2})
= a_{\delta_2+\delta'_2} a_{\delta_2+\delta'_2} a_{\delta_2+\delta'_2} a_{\delta_2+\delta'_2} a_{\delta_2+\delta'_2}.
\end{align*}

Thus we can finish our proof of Theorem 2. \hfill \Box

**Remark 2.** When $G = \text{GL}(n, C)$, irreducible finite dimensional representations of $K = \text{U}(n)$ are explicitly described in terms of Gel'fand-Tsetlin basis. If $\pi$ is the principal series representation of $G$, we may take a pair $(f_1, f_2, \Phi)$ in a similar way as above. Instead of $c_{\delta_2}^{\delta_1}$ in $\Phi$, certain Clebsch-Gordan coefficients appear. We want to discuss this case in a future paper.

**References**

